DFG-GK 1821
"Cohomological methods in Geometry"

The geometry of secant varieties to algebraic curves

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Motivation

The geometry of secant varieties appears under various guises while solving enumerative problems for curves or higher dimensional varieties:

- The number of nodes of a curve contained in the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$ can be obtained by counting the points in the intersection of two secant varieties to the curve.
- The number of certain secant varieties to a projective surface can be obtained via integrals of Segre classes of tautological line bundles over the Hilbert scheme of points of the surface, which also appear in Donaldson-Thomas counting of sheaves.

It is worthwhile to better understand the (enumerative) geometry of secant varieties and their intersections!

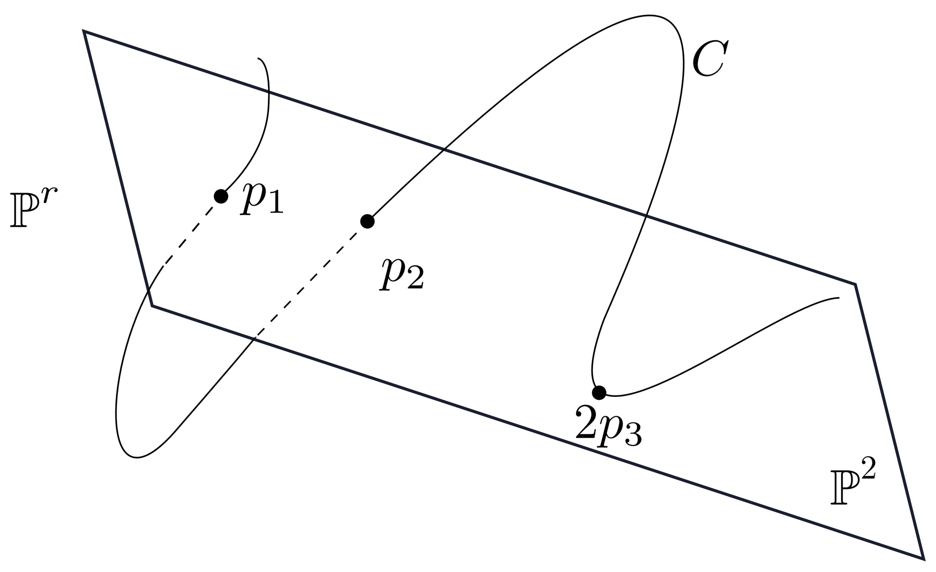
Secant varieties

We focus on secant varieties of curves.

Let C be a smooth curve of genus g and l a linear series of degree d and dimension r. Fix $0 \leq f < e \leq d \in \mathbb{N}$. The **secant variety**

$$V_e^{e-f}(l) = \{D \in C_e \mid \dim(l - D) \geq r - e + f\} \subset C_e$$

parametrises e-secant $(e-f-1)$-planes to C inside $\mathbb{P}^r$.



A 4-secant 2-plane to C in $\mathbb{P}^r$

$V_e^{e-f}(l)$ is a degeneracy locus in C_e and has expected dimension

$$\exp\dim V_e^{e-f}(l) = e - f(r + 1 - e + f).$$

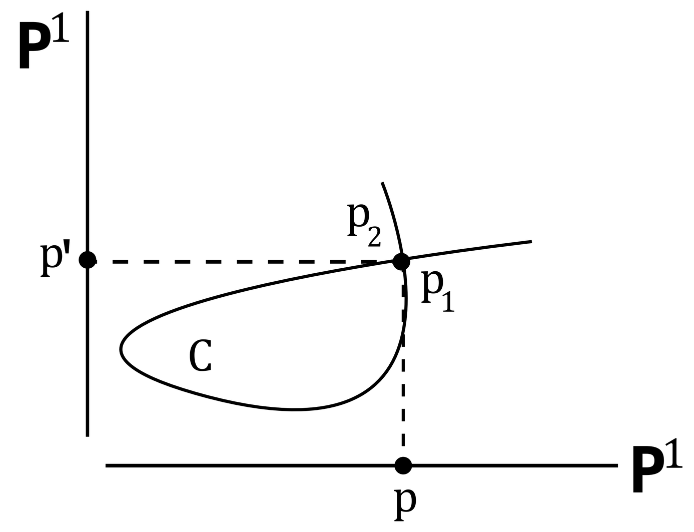
Let $G_d^r(C)$ parametrise all linear series of degree d and dimension r on C. We know: if C is general, then $G_d^r(C)$ is smooth, reduced and of expected dimension.

Farkas [1] showed:

- If C is general and $\dim G_d^r(C) + \exp\dim V_e^{e-f}(l) < 0$, then
$$V_e^{e-f}(l) = \emptyset \quad \forall l \in G_d^r(C).$$
- If C and $l \in G_d^r(C)$ are general and $V_e^{e-f}(l) \neq \emptyset$, then
$$\dim V_e^{e-f}(l) = \exp\dim V_e^{e-f}(l) \geq 0.$$
- If C is general and some numerical conditions are satisfied, then:
$$\exp\dim V_e^{e-f}(l) \geq 0 \Rightarrow V_e^{e-f}(l) \neq \emptyset. \tag{1}$$

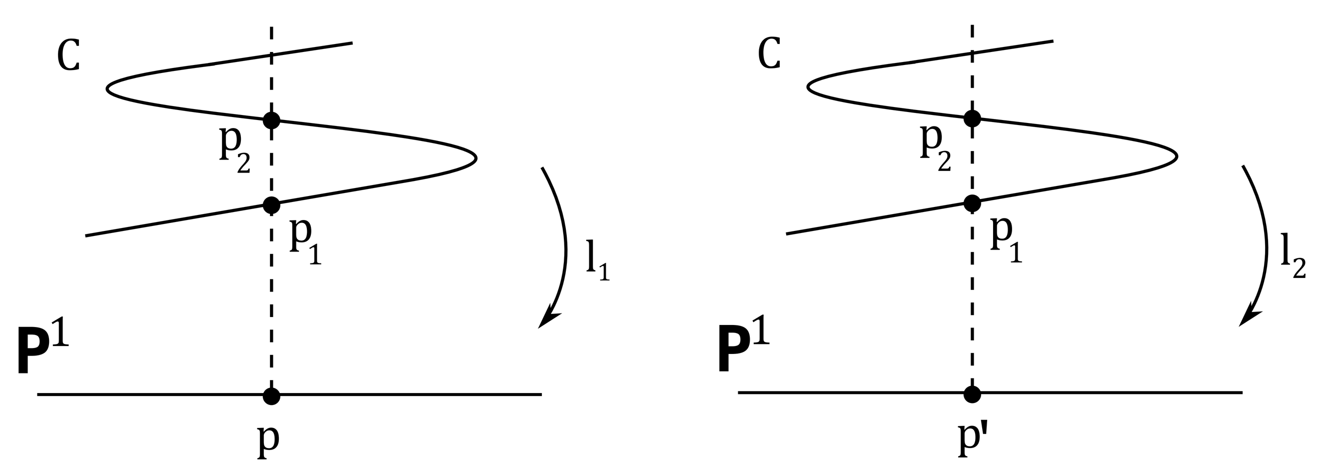
Intersections of secant varieties

Example: C curve of bidegree (d_1, d_2) in $\mathbb{P}^1 \times \mathbb{P}^1$ with node given by pair of points (p_1, p_2), where $p_1, p_2 \in C$:



Can reformulate: C is equipped with two linear series

- $l_1 \in G^1_{d_1}(C)$ with $\dim(l_1 - p_1 - p_2) \geq 0$,
- $l_2 \in G^1_{d_2}(C)$ with $\dim(l_1 - p_1 - p_2) \geq 0$:



The node is a divisor $p_1 + p_2 \in V_2^1(l_1) \cap V_2^1(l_2) \in C_2$ and we expect finitely many nodes because $\exp\dim V_2^1(l_1) + \exp\dim V_2^1(l_2) = 2 = \dim C_2$.

One computes number of nodes to be $(d_1 - 1)(d_2 - 1) - g$.

More generally: if $l_1 \in G^{r_1}_{d_1}(C)$, $l_2 \in G^{r_2}_{d_2}(C)$, and $\exp\dim V_e^{r_1}(l_1) + \exp\dim V_e^{r_2}(l_2) = e = \dim C_e$, then the expected number of points in $V_e^{r_1}(l_1) \cap V_e^{r_2}(l_2)$ is ([2]) the coefficient of $t_1^{e-r_1} t_2^{e-r_2}$ in

$$(1 + t_1)^{d_1 - g - r_1}(1 + t_2)^{d_2 - g - r_2}(1 + t_1 + t_2)^g. \tag{2}$$

Unexpected behaviour

The formula (2) above yields unexpected zero counts when $l_2 = K_C - l_1$!

We investigated the zero-count situation and found [3]:

- Cases when $V_e^{r_1}(l_1) \cap V_e^{r_2}(l_2)$ is empty,
- Sufficient conditions for $V_e^{r_1}(l_1) \cap V_e^{r_2}(l_2)$ to be positive-dimensional.

We also found a counterexample to the expectation of non-emptiness of secant varieties (1): for $l \in G^{d-3}_{3d-8}(C)$ we have $V^{d-4}_{2d-8}(l) = \emptyset$ for $d \geq 4$ although $\exp\dim V^{d-4}_{2d-8}(l) = 0$.

References

[1] G. Farkas. *Higher ramification and varieties of secant divisors on the generic curve.* Journal of the LMS, 78:418-440, 2008.

[2] E. Arbarello, M. Cornalba, P.Griffiths, J. Harris. *Geometry of Algebraic Curves.* Springer, 1985.

[3] M. Ungureanu *Geometry of intersections of some secant varieties to algebraic curves.* arXiv:1810.05461 [math.AG].