A log-tropical criterion for existence of Néron models

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Good and bad reduction of abelian varieties

Consider a discrete valuation ring $(R, \mathfrak{m})$, with fraction field K and residue field k, and an abelian variety A defined over K. One says that A has **good reduction** if there exists a smooth and proper $\mathcal{A}/R$ extending A.
In this case, $\mathcal{A}/R$ is automatically an abelian scheme, and by properness there is a reduction mod $\mathfrak{m}$ homomorphism $A(K) = \mathcal{A}(R) \to \mathcal{A}_k(k)$.
One says that A has **bad reduction** if it has not good reduction. In this case, one may still want to find a canonical way to extend A to a model $\mathcal{A}/R$.

Néron models

Néron models are a possible answer to the above problem. A Néron model $\mathcal{N}/R$ for A is a scheme together with an isomorphism $\mathcal{N} \times_R K = A$, enjoying a number of pleasant properties. Here is a list of the best ones:

- it exists;
- it is unique up to unique isomorphism;
- it is smooth;
- it has a unique group structure inherited from A;
- it has a reduction mod $\mathfrak{m}$ homomorphism $A(K) \to \mathcal{N}_k(k)$;

Here is one bad property: the formation of Néron model does not commute with base change (however, it does commute with extensions of DVR's $R \to R'$ of ramification index 1). In general, the closed fibre $\mathcal{N}_k$ of the Néron model is disconnected.

Example 1

Let E be an elliptic curve over $\mathbb{Q}$, and p a prime. Let $\mathcal{E}$ be the minimal regular model of E over $\mathbb{Z}_{(p)}$. Then

$$\mathcal{N} = \mathcal{E}^{sm}$$

the $\mathbb{Z}_{(p)}$-smooth locus of $\mathcal{E}$. The group of components of $\mathcal{N}_{\mathbb{F}_p}$ is cyclic of order equal to the valuation of the j-invariant of E.

Bases of higher dimension

One can ask if a theory of Néron model exists when the discrete valuation ring R is replaced by a scheme S of dimension higher than one. Namely:

Question 1

Let S be a regular, noetherian scheme, $U \subset S$ an open dense, A/U an abelian scheme. Does there exist a Néron model $\mathcal{N}/S$ for A over S?

The answer, given by David Holmes, is: in general, **no**. The following is the most basic counterexample.

Example 2: [Hol17]

Let C_0 be the so called **banana curve** over $k = \bar{k}$: the union of two $\mathbb{P}^1$'s intersecting transversally at two points. Fix two smooth points P, Q on the two components, and let $(\mathcal{C}/S, \sigma_1, \sigma_2)$ be the universal deformation of C_0 as a 2-marked genus 1 stable curve. Now forget the section σ_2. Generically on S, we have an elliptic curve E, which does not admit a Néron model over S.

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A criterion for existence

Let S be a complete, local ring, $D = D_1 \cup D_2 \cup \ldots \cup D_n \subset S$ a normal crossing divisor, $U = S \setminus D$ its open dense complement. Let $\mathcal{C}/S$ be a nodal curve, smooth over U, and $Pic^0_{\mathcal{C}_U/U}$ the jacobian.
Let Γ be the dual graph of the closed fibre $\mathcal{C}_0$, and $\Gamma_1, \ldots, \Gamma_n$ the dual graphs at the generic points of $D_1, \ldots, D_n$ respectively.
Since each Γ_i is a contraction of Γ, there is a natural map

$$p\colon H_1(\Gamma, \mathbb{Z}) \hookrightarrow \bigoplus_{i=1}^{n} H_1(\Gamma_i, \mathbb{Z})$$

Definition 1

We say that $\mathcal{C}/S$ is toric additive if p is an isomorphism.

Recently, in [MW18], Molcho and Wise have introduced a new notion of logarithmic jacobian and tropical jacobian of a log smooth curve. These are sheaves of abelian groups on $S_{ét}$, and are in general not representable.

Theorem 1

The following are equivalent for $\mathcal{C}/S$:

- it is toric additive;
- the jacobian $\mathrm{Pic}^0_{\mathcal{C}_U/U}$ admits a Néron model over S;
- the logarithmic jacobian is representable and $\mathrm{LogPic}^0_{\mathcal{C}/S}$ is a Néron model for $\mathrm{Pic}^0_{\mathcal{C}_U/U}$;
- the tropical jacobian $\mathrm{TropJac}_{\mathcal{C}/S}$ is representable and takes values in **finite** abelian groups.

When the conditions are satisfied, we have an exact sequence

$$0 \to \mathrm{Pic}^0_{\mathcal{C}/S} \to \mathcal{N} \to \mathrm{TropJac}_{\mathcal{C}/S} \to 0$$

In particular, $\mathrm{TropJac}_{\mathcal{C}/S}$ is the group of components of the Néron model.

Example 3

If $\mathcal{C}/S$ has fibres of compact-type (i.e. all dual graphs are trees), then it is automatically toric additive. Similarly, if $D = D_1$ is a regular divisor.

Conjecture 1

The curve $\mathcal{C}/S$ is toric additive if and only if $\mathrm{LogPic}^0_{\mathcal{C}/S}$ and $\mathrm{TropJac}_{\mathcal{C}/S}$ are representable by algebraic spaces.

Abelian schemes

If we replace $\mathcal{C}/S$ by a semiabelian scheme $\mathcal{A}/S$, whose restriction to U is abelian, the definition of toric additivity generalizes (by replacing homology groups by character groups of tori).

Theorem 2

Suppose S has equal characteristic zero. If $\mathcal{A}/S$ is toric additive, then $\mathcal{A}_U$ admits a Néron model over S.

I don't know yet if the converse is true.

References

[Hol17] David Holmes. Néron models of jacobians over base schemes of dimension greater than 1. *To appear in Journal für die reine und angewandte Mathematik*, 2017.

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